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$p=a\sqrt{3}\cos\beta=a\sqrt{3}\,(\cos\psi'\cos\theta+\sin\psi'\sin\theta\cos\alpha)$
$=a[\cos\theta(\cos\varphi+\sin\varphi)+\sin\theta]$.

The total number of cases is

$$T=4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{p'}\cos\theta\, dp\, d\theta\, d\varphi=3\pi a.$$

Therefore the probability is $P=E/T=4/\pi(\sqrt{3}\tan^{-1}\frac{1}{3}\sqrt{3}-\sqrt{2}\tan^{-1}\frac{1}{2}\sqrt{2})$.

Note.—I wish to acknowledge my indebtedness to Professor DeLong and Mr. Frank Giffin for looking over this solution and making valuable suggestions.

MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Prove $\tan^{-1}x=\dfrac{1}{2i}\left(\log\dfrac{x-i}{x+i}\right)$, and thence that $\pi=(2/i)\log(i)$.

I. Solution by GUY B. COLLIER and HAROLD C. FISKE, Class 1901, Union College, Schenectady, N. Y., and the PROPOSER.

Consider the integral,

$$\int\frac{dx}{1+x^2}=\tan^{-1}x\ldots\ldots(1).$$

Integrate the left member by partial fractions

$$\int\frac{dx}{1+x^2}=\frac{1}{2i}\int\frac{dx}{x-i}-\frac{1}{2i}\int\frac{dx}{x+i}=\frac{1}{2i}\log\frac{x-i}{x+i}\ldots\ldots(2).$$

∴ From (1) and (2),

$$\tan^{-1}x=\frac{1}{2i}\log\frac{x-i}{x+i}.$$

When $x=1$ this becomes

$$\tfrac{1}{4}\pi=\frac{1}{2i}\log\frac{1-i}{1+i}=\frac{1}{2i}\log(-i).$$

$\pi=(2/i)\log(-i)$, it should have been.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

We have the identity

$$\frac{1}{1+x^2}=\frac{1}{2i}\left(-\frac{1}{i-x}-\frac{1}{i+x}\right).\ \therefore\int\frac{dx}{1+x^2}=\frac{1}{2i}\log\left(\frac{i-x}{i+x}\right)+c.$$

$$\text{But}\int\frac{dx}{1+x^2}=\tan^{-1}x+c.\quad\therefore\ \tan^{-1}x=\frac{1}{2i}\log\left(\frac{i-x}{i+x}\right).$$

Making $x=0$, it is seen that no constant need be added.

Now making $x=1$, we have

$$\tfrac{1}{4}\pi=\frac{1}{2i}\log\left(\frac{i-1}{i+1}\right)=\frac{1}{2i}\log\left(\frac{-2i}{-2}\right)=\frac{1}{2i}\log(i)\qquad\therefore\ \pi=\frac{2}{i}\log(i).$$

Or we may obtain this from the equation $e^{i\theta}=\cos\theta+i\sin\theta$ by putting $\theta=\frac{1}{2}\pi$ and taking logarithm of both sides.

(The proposer of this problem seems to have neglected the matter of determining the constant and so has the sign under the logarithm wrong.)

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$$\frac{1}{1+x^2}=1/2i.\frac{(x+i)-(x-i)}{1+x^2}=1/2i\frac{[(x+i)-(x-i)](x+i)}{(x+i)^2(x-i)}.\quad\text{When } i=\sqrt{(-1)}.$$

$$\therefore\ \frac{dx}{1+x^2}=1/2i.\frac{[(x+i)-(x-i)]\,(x+i)dx}{(x+i)^2(x-i)}.\quad\therefore\ \tan^{-1}x=1/2i.\left(\log\frac{x-i}{x+i}\right).$$

Let $x=1$. $\therefore\ \frac{1}{4}\pi=(1/2i)\log\dfrac{1-i}{1+i}=(1/2i)\log(-1)$. $\therefore\ \pi=(2/i)\log(-i)$.

IV. Solution by J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

Let $y+ix=re^{ia}$ ..... (1), and $y-ix=re^{-ia}$ ......(2).

Then $r^2=x^2+y^2$, $\tan a=x/y$.

From (1) and (2), $\log(y+ix)-\log(y-ix)=2ia=2i\tan^{-1}(x/y)$, or

$$\log\frac{y+ix}{y-ix}=2i\tan^{-1}(x/y);\ \text{whence}\ \log\left(\frac{(y/i)+x}{(-y/i)+x}\right)=2i\tan^{-1}(x/y),$$

$$\text{or}\ \log\left(\frac{x-iy}{x+iy}\right)=2i\tan^{-1}(x/y).$$

Put $y=1$, then $(1/2i)\log\dfrac{x-i}{x+i}=\tan^{-1}x$.

If $x=0$, then $\tan^{-1}0=\pi=(1/2i)\log(-i/i)$, or $\pi=(1/2i)\log(i)$.

V. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting in $\log(a+bi)$, $a=\rho\cos\phi$, $b=\rho\sin\phi$ .....(1).

Whence $\rho=(a^2+b^2)^{\frac{1}{2}}$ ..... (2), $\phi=\tan^{-1}(b/a)$ .....(3).

We have $\log(a+bi)=\log\rho+\log(\cos\phi+\sin\phi\sqrt{-1})=\log\rho+\log e^{\phi\sqrt{-1}}$

$$=\log\rho+\phi\sqrt{-1}=\tfrac{1}{2}\log(a^2+b^2)+i\tan^{-1}(b/a).$$

$\therefore\ \tan^{-1}(b/a)=(1/i)[\log(a+bi)-\tfrac{1}{2}\log(a^2+b^2)]$

$$=(1/i)\log\frac{a+bi}{(a^2+b^2)^{\frac{1}{2}}}=(1/2i)\log\frac{(a+bi)^2}{a^2+b^2}=(1/2i)\log\frac{a+bi}{a-bi}$$

sin$a^2+b^2=(a+bi)(a-bi)$.

Putting now $b=x$, $a=1$, we have

$$\tan^{-1}x=(1/2i)\log\frac{1+ix}{1-ix},$$

or multiplying numerator and denominator of the fraction under the logarithmic function by i, we have

$$\tan^{-1}x=(1/2i)\log\frac{i-x}{i+x}.$$

Putting $x=1$, we have $\tfrac{1}{4}\pi=(1/2i)\log\dfrac{i-1}{i+1}=(1/2i)\log\dfrac{(i-1)^2}{(i+1)(i-1)}$

$$=(1/2i)\log(-2i/-2)=(1/2i)\log i.\quad \therefore\ \pi=(2/i)\log i.$$

Note.—There is a slight error in the statement, since $x-i$ should be $i-x$.

VI. Solution by COOPER D. SCHMITT, A.M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn., and ELMER SCHUYLER, High Bridge, N. J.

From trigonometry we have the formula

$$\log[\alpha+\beta i)=\log\sqrt{(\alpha^2+\beta^2)}+i\tan^{-1}(\beta/\alpha).$$

Let $\alpha=x$, $\beta=-1$, and we have

$$\log(x-i)=\log\sqrt{(x^2+1)}+i\tan^{-1}(-1/x).$$

Let $\alpha=x$, $\beta=1$, and we have

$$\log(x+i)=\log\sqrt{(x^2+1)}+i\tan^{-1}(1/x).$$

Subtracting, we have

$$\log\frac{x-i}{x+i}=i[\tan^{-1}(-1/x)-\tan^{-1}(1/x)=i\tan^{-1}\left(\frac{-(1/x)-(1/x)}{1-(1/x^2)}\right)$$

$$=i\tan^{-1}\frac{2x}{1-x^2}=2i\tan^{-1}x.$$

Now let $x=1$, and we have

$$2i\tfrac{1}{4}\pi=\log\frac{1-i}{1+i}=\log i,\ \text{or}\ i\pi=2\log i,\ \pi=(2/i)\log i.$$

Or further, $i\pi=\log i^2=\log(-1)$, or $\pi=\dfrac{\log(-1)}{\sqrt{(-1)}}$.